Characterization of Quasi-Stationary Distributions

Iddo Ben-Ari (University of Connecticut), joint with Ningwei Jiang

RossFest70, Saint Louis University

Setup

Assumption 1

Let $\mathbf{X} = (X_t : t \in \mathbb{Z}_+)$ be MC on a countable state space $S \cup \{\Delta\}$ w/transition function p.

1. *Δ is a unique absorbing state: $p(\Delta, \cdot) = \delta_\Delta(\cdot)$.*
2. *The restriction of p to $S \times S$ is irreducible.*

Let τ_Δ denote the absorption time:

$$\tau_\Delta = \inf\{t \in \mathbb{N} : X_t = \Delta\}.$$

3. *There exists $\beta > 0$ such that $E_x[\exp(\beta \tau_\Delta)] < \infty$ for some (equivalently all) $x \in S$.*

Definition 1 (QSD)

A probability measure ν on S is a Quasistationary Distribution if

$$P_\nu(X_t \in \cdot \mid \tau_\Delta > t) = \nu(\cdot)$$

for all $t \in \mathbb{Z}_+$.

Basic Properties

Proposition 1

- *(Necessary condition) If ν is a QSD, then under P_ν, τ_Δ has a geometric distribution with parameter $1 - e^{-\lambda}$ for some $\lambda > 0$.*

$$P_\nu(\tau_\Delta > t) = e^{-\lambda t}, \ t \in \mathbb{Z}_+. \tag{1}$$

  - *λ is called the absorption parameter for ν.*
- *(Eigenvector) A probability measure ν on S is a QSD with absorption parameter λ if and only if it is an $e^{-\lambda}$-invariant probability measure for p:*

$$\nu p = e^{-\lambda} \nu.$$

  - *Equivalently, ν satisfies the non-linear eigenvalue equation:*

$$\nu p = ((\nu p)\mathbf{1})\nu.$$

Note. (1) explains why we assume finiteness of exponential moments for τ_Δ and is useful in proving non-existence.

Proposition 2 (Quasi-limiting ⇒ QSD)

A probability measure ν on S is a QSD if and only if

$$\lim_{t\to\infty} P_\mu(X_t \in \cdot \mid \tau_\Delta > t) = \nu \text{ for some initial distribution } \mu.$$

A Critical Parameter

Definition 2 (Critical Absorption Parameter)

- *Let*
$$\lambda_{cr} = \sup\{\lambda > 0 : E_x[\exp(\lambda\tau_\Delta)] < \infty \text{ for some } x \in S\}. \quad (2)$$
In words: the tail of τ_Δ decays geometrically like $t \to e^{-\lambda_{cr} t}$ under P_x for every $x \in S$.
- *A QSD with absorption parameter λ_{cr} is called minimal.*

Corollary 3

1. $\lambda_{cr} \in (0, \infty)$.
2. *If ν is a QSD with absoprtion parameter λ, then $\lambda \leq \lambda_{cr}$.*

Regimes for QSDs

Regimes Identified

Existence, Uniqueness and representation of QSDs are according to the following.

$$
\begin{tikzcd}
 & \text{Infinite MGF} \\
E_x[\exp(\lambda\tau_\Delta)] \arrow[ur, "{=\infty \ (\&\ \lambda=\lambda_{cr})}"] \arrow[dr, "<\infty"'] & \\
 & \text{Finite MGF}
\end{tikzcd}
$$

- Infinite MGF (only applicable to λ_{cr}):
  - "Perron Frobenius": Reminiscent to positive recurrent MCs.
  - Related works include: Seneta and Vere-Jones (1966), Ferrari, Kesten and Martinez (1996).
- Finite MGF
  - QSDs are completely characterized by Martin entrance boundaries indexed by λ and represented through a Choquet-type theorem.
  - Construction of Martin boundary specific to this application, directly from pointwise limits of **normalized** Green's functions (no need for additional states, or analysis through harmonic functions of dual process).
  - λ_{cr} may be in this regime.

Infinite MGF Regime

For $x \in S$ let

$$\tau_x = \inf\{t \in \mathbb{N} : X_t = x\}.$$

Theorem 4

Suppose λ_{cr} is in the infinite MGF regime. Then there exists a minimal QSD if and only if

$$E_x[\exp(\lambda_{cr}\tau_\Delta), \tau_\Delta < \tau_x] < \infty \text{ for some } x \in S. \tag{3}$$

In this case, the minimal QSD is unique and is given by

$$\nu_{cr}(x) = \frac{e^{\lambda_{cr}} - 1}{E_x[\exp(\lambda_{cr}\tau_\Delta), \tau_\Delta < \tau_x]} \tag{4}$$

Note.

- Stronger assumptions guarantee existence of "Yaglom Limit": ν_{cr} as the quasi-limit from any finitely supported initial distribution.
- Can be useful for simulations of minimal QSDs.
- Denominator is also equal to $E_x[\exp(\lambda_{cr}\tau_\Delta \wedge \tau_x)] - 1$. Limit as $\lambda_{cr} \to 0$?

Examples

- Finite S. In this case, ν_{cr} is a unique QSD (no other absorption rates).
- (Under standard assumptions): Subcritical branching. Here $e^{\lambda_{cr}} = \frac{1}{m}$ where m is the expectation of the offspring distribution.

Special Case: Coming Fast From Infinity

For $K \subseteq S$ let

$$\tau_K = \inf\{t \in \mathbb{N} : X_t \in K\}.$$

The following was inspired and implies (a special case) of the main result of Martinez, Martin, and Villemonais (2014):

Theorem 5

Suppose that there exists some $\bar{\lambda} > 0$ and a nonempty finite $K \subsetneq S$ such that:

$$E_x[\exp(\bar{\lambda}\tau_\Delta)] = \infty \text{ for some } x \in S;$$

$$\sup_{x \notin K} E_x[\exp(\bar{\lambda}\tau_\Delta \wedge \tau_K)] < \infty.$$

Then $\lambda_{cr} \in (0, \bar{\lambda}]$, and the conditions on Theorem 4 hold, ν_{cr} is the unique QSD, and is the quasi-limiting distribution from any finitely supported initial distribution.

Note.

- Under additional assumptions: ν_{cr} is the quasi-limiting distribution from any initial distribution.
- The cited work did not assume irreducibility and does not provide a concrete formula for the stationary distribution.

Finite MGF Regime

Assumption. S is infinite.

The following is a slight generalization of Ferrari, Kesten, Martinez and Picco (1995):

Theorem 6

Let $\lambda > 0$ be in the finite MGF regime.

1. *If $\lim_{x\to\infty} E_x[\exp(\lambda'\tau_\Delta)] = \infty$ for some $\lambda' \in (0,\lambda)$, then there exists a QSD w/absorption parameter λ.*
2. *If $\limsup_{x\to\infty} E_x[\exp(\lambda\tau_\Delta)] < \infty$, then there does not exist a QSD w/ absorption parameter λ.*

Corollary 7 (Continuum of QSDs)

Let

$$\lambda_0 = \inf\{\lambda > 0 : \lim_{x\to\infty} E_x[\exp(\lambda\tau_\Delta)] = \infty\}.$$

Then for every $\lambda \in (\lambda_0, \lambda_{cr}]$ there exists a QSD w/absorption parameter λ.

Note. The existence of a minimal QSD requires a proof.

Examples

Immediate applications of the Corollary with $\lambda_0 = 0$.

- Birth & Death Chain on $\mathbb{Z}_+ \cup \{-1\}$ with absorption at $\Delta = -1$, satisfying our assumptions.
- (Under standard assumptions): Subcritical branching.

Finite MGF Regime: Martin Boundary

Overview

- Classically, Martin exit / entrance boundaries are used to characterize harmonic functions / invariant measures.
- Our work applies this for the specific task of classifying QSDs w/absorption parameter λ rather than the more general class of e^{λ}-invariant measures (which may be infinite).
- How exactly? One minor observation. In the finite MGF regime Green's functions are normalizable allowing to construct a Martin boundary as limits of probability measures.

Definition 3

Let $\lambda > 0$ be in the finite MGF regime.

- *Green's function.* $G^{\lambda}(x,y) = \delta_x(y) + \sum_{n=1}^{\infty}(e^{\lambda}p)^n(x,y) = \frac{E_x[\exp(\lambda\tau_y),\tau_y<\tau_\Delta]}{1-E_y[\exp(\lambda\tau_y),\tau_y<\tau_\Delta]}$.
- *Normzlized Green's Kernel.* $K^{\lambda}(x,y) = \frac{G^{\lambda}(x,y)}{G^{\lambda}(x,\mathbf{1})}$, *where* $G^{\lambda}(x,\mathbf{1}) = \sum_{y\in S} G^{\lambda}(x,y) = G^{\lambda}(x,\mathbf{1}) = \frac{E_x[\exp(\lambda\tau_\Delta)]-1}{e^{\lambda}-1} < \infty$.

Next slides. Compactification M^{λ} of S so that the function $M^{\lambda} \ni x \to K^{\lambda}(x,\cdot)$ is continuous.

Martin Compactification

Definition 4 (Martin Compactification)

- *A sequence* $\mathbf{x} = (x_n : n \in \mathbb{N})$ *in* S *satisfying* $\lim_{n\to\infty} x_n = \infty$ *is* λ*-convergent if the point-wise limit* $\lim_{n\to\infty} K^\lambda(x_n, \cdot)$ *exists.*
- *Two* λ*-convergent sequences* $\mathbf{x}$ *and* $\bar{\mathbf{x}}$ *are equivalent if*

$$\lim_{n\to\infty} K^\lambda(x_n, y) = \lim_{n\to\infty} K^\lambda(\bar{x}_n, y) \text{ for all } y \in S.$$

- *Write* $[\mathbf{x}]$ *for the equivalency class of the convergent sequence* $\mathbf{x}$*.*
- *Martin Boundary. Let*

$$\begin{aligned} K^\lambda([\mathbf{x}], \cdot) &= \lim_{n\to\infty} K^\lambda(x_n, \cdot) && \leftarrow \textit{ boundary points} \\ \partial^\lambda M &= \{[\mathbf{x}] : K^\lambda([\mathbf{x}], \cdot)\} && \leftarrow \textit{ Martin Boundary} \\ M^\lambda &= S \cup \partial^\lambda M && \leftarrow \textit{ Martin Space} \end{aligned}$$

- *Metric: For* $a, b \in M^\lambda$*, let*
$$\rho^\lambda(a, b) = \sum_{n=1}^{\infty} \frac{1}{2^n} \Big(|\delta_{a,s_n} - \delta_{b,s_n}| + d(K^\lambda(a, s_n), K^\lambda(b, s_n)) \Big),$$ *where* $(s_n : n \in \mathbb{N})$ *are the distinct elements of* S *listed as a sequence and* $d(i, j) = \frac{|i-j|}{1+|i-j|}$*.*

Martin Compactification: Properties

Proposition 8 (Properties of the metric space)

- $(M^\lambda, \rho^\lambda)$ *is a compact metric space and* $\partial^\lambda M$ *is closed.*
- *A sequence* $\mathbf{a} = (a_n : n \in \mathbb{N})$ *of elements of* M^λ *is* ρ^λ *convergent if and only if either*
  1. *There exists* $a \in \mathbb{N}$ *and* $n_0 \in \mathbb{N}$ *such that* $a_n = a$ *for all* $n \geq n_0$*:*
     - $\lim_{n\to\infty} a_n = a$*; or*
  2. *Condition 1 does not hold and there exists* $[\mathbf{a}] \in \partial^\lambda M$ *such that* $\lim_{n\to\infty} K^\lambda(a_n, \cdot) = K^\lambda([\mathbf{a}], \cdot)$
     - $\lim_{n\to\infty} a_n = [\mathbf{a}]$.

Let

$$S^\lambda = \{[\mathbf{x}] \in \partial^\lambda M : K^\lambda([\mathbf{x}], \cdot) \text{ is a QSD with absorption parameter } \lambda\}.$$

Theorem 9 (Choquet Representation)

Let $\lambda > 0$ *be in the finite MGF regime. Then.*

- *There exists a QSD with absorption parameter* λ *if and only if* S^λ *is not empty.*
- *In this case,* ν *is a QSD with absorption parameter* λ *if and only if there exists a Borel probability measure* $\bar{F}_\nu$ *on* M^λ *satisfying* $\bar{F}(S^\lambda) = 1$ *and*

$$\nu(y) = \int K^\lambda([\mathbf{x}], y) d\bar{F}_\mu([\mathbf{x}]),\ y \in S.$$

Continuous Time

Simply put

All assumptions and results have the direct analogs in the continuous-time setting

Notation

- We denote the process by $\mathbb{X} = (\mathbb{X}_t : t \in \mathbb{R}_+)$.
- Transition rate from $x \in S$ is $q_x \in (0, \infty)$.
- For $x \in S \cup \{\Delta\}$ we write τ_x for the hitting time of x,

$$\tau_x = \inf\{t \in \mathbb{R}_+ : \mathbb{X}_t = x \text{ and } \mathbb{X}_{t-} \neq x\}.$$

- We denote the critical absorption parameter by λ_{cr}.

Continuous Time Example: Birth & Death

Notation

- $S = \mathbb{Z}_+ \cup \{-1\}$ with $\Delta = -1$.
- As usual λ_n and μ_n are the birth and death rates from n, respectively.
- Let s be the absorption time starting from ∞ (well defined due to stochastic domination).
- For the record:

$$E[s] = \sum_{n=1}^{\infty} \frac{1}{\lambda_n \pi_n} \sum_{i=n+1}^{\infty} \pi_i, \text{ where } \pi_0 = 1, \pi_n = \prod_{j=1}^{n} \frac{\lambda_{j-1}}{\mu_j}, \ n \in \mathbb{N}.$$

Our results give a simple and short proof to the following through applications of both infinite and finite MGF regimes:

Theorem 10 (Van Doorn (1991), Theorem 3.2)

1. *Suppose $E[s] < \infty$. Then $\mathbb{\lambda}_{cr} > 0$, and there exists a unique QSD, which is also minimal.*
2. *Suppose that $E[s] = \infty$. Then either $\mathbb{\lambda}_{cr} = 0$ and there are no QSDs or $\mathbb{\lambda}_{cr} > 0$ and for every $\lambda \in (0, \mathbb{\lambda}_{cr}]$ there exists a QSD with absorption parameter λ.*
3. *When exists, a QSD with absorption parameter $\lambda > 0$ is unique and given by the formula*

$$\nu_\lambda(y) = \frac{\lambda}{q_y - \lambda} \frac{1}{E_y[\exp(\lambda \tau_\Delta), \tau_\Delta < \tau_y]}, \ y \in S. \tag{5}$$

Happy 70, Ross!